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Early Universe with Variable Cosmological Constants in Higher Dimensional Space Time

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Abstract The field equations with variable cosmological and gravitational constants are consider in the presence of perfect fluid for Kaluza-Klein type cosmological model. The exact solutions of the field equations are obtained by using the gamma law equation of state $p = (\gamma - 1)\rho$ in which the parameter γ depends on scale factor R. The functional form of $\gamma(R)$ is used to analyze a wide range of cosmological solution at early universe for two phases in cosmic history: *inflationary phase* and *radiation dominated phase*. The corresponding physical interpretation of cosmological solution are also discussed in the framework of higher dimensional space time.

Keywords Early universe · Gravitational and cosmological constants

1 Introduction

The Kaluza-Klein theory [1, 2] is striking because it has a particularly elegant presentation in terms of geometry. In a certain sense, it looks just like ordinary gravity in free space, except that it is phrased in five dimensions instead of four. A number of authors [3–5] have studied physics of the universe in higher-dimensional space-time.

Overduin and Wesson [6] have presented an excellent review of Kaluza-Klein theory and higher-dimensional unified theories, in which the cosmological and astrophysical implications of extra-dimension have been discussed. Also, many authors have studied Kaluza-Klein cosmological models with different matters [7–11]. There is now extensive literature

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dealing with different aspect of higher dimensional cosmologies. Some authors explain entropy production [12, 13] and inflation [14, 15] as a results of contraction of extra space. A great number of exact cosmological solutions of Einstein fields equations with different equation of state and different symmetries, including or not a cosmological constant, has been found with 5D [16–18] and also with arbitrary number of dimension [19–22].

In relativistic and observational cosmology, the evolution of the universe is described by Einstein's field equations together with perfect fluid and an equation of state. Einstein's theory of gravity contains two parameters-Newtonian gravitational 'constant' G and cosmological 'constant' Λ. Normally, these are considered as fundamental constants. The gravitational 'constant' G plays the role of a coupling constant between geometry of space and matter content in Einstein field equations. In an evolving universe, it appears natural to look at this constant as a function of time. Dirac [23] and Dicke [24] have suggested a possible time varying gravitational constant. The Large Number Hypothesis (LNH) proposed by Dirac [25, 26] lead to a cosmology where G varies with the cosmic time. There have been many extensions of Einstein's theory of gravitation, with time-dependent G, in order to achieve a possible unification of gravitation and elementary particle physics. Canuto et al. [27] made numerous suggestions based on different arguments that G is indeed time dependent.

On the other hand, there exist several interesting physical problems, where the Einstein's field equations for homogeneous, isotropic, and spatially flat cosmological models with and without cosmological 'constant' Λ with varies matter sources, reduce to particular cases of the second-order nonlinear ordinary differential equations. The Λ-term arises naturally in general-relativistic quantum field theory where it is interpreted as the energy density of the vacuum (see [28]). It is widely believed that the value of Λ was large during the early stages of evolution and strongly influenced its expansion, whereas its present value is incredibly small. The possibility of Λ as a function of time has been considered by Canuto et al. [29].

It is still a challenging problem to know the exact physical situation at very early stages of the formation of our universe. At the very early stages of evolution of universe, it is generally assumed that during the phase transition (as the universe passes through it's critical temperature) the symmetry of the universe is broken spontaneously. Carvalho [30] and Singh [31] have studied Robertson-Walker model in general relativity by using equation of state $p = (\gamma - 1)\rho$, where the adiabatic parameter γ varies with cosmic time. A unified description of early evolution of universe has been presented in which an inflationary phase is followed by a radiation dominated period. His work motivate one to consider further work in some alternative theories of gravitation.

In this paper we consider the Kaluza-Klein type cosmological model with variable G and Λ in the presence of perfect fluid. Our approach is similar to that of Carvalho [30] but with time dependent gravitational and cosmological constants. We study the evolution of the universe as it goes from inflationary phase to radiation dominated phase in the context of higher dimensional space time. The paper is organized as follows: In Sect. 2 we present the basic field equations of Kaluza-Klein type models. In Sect. 3 we discuss the solutions of the field equations for different early phases: *inflationary* and *radiation dominated*. The conclusion are discussed in Sect. 4.

2 Model and Field Equations

Let us consider the Kaluza-Klein type metric

$$ds^2 = dt^2 - R^2(dx^2 + dy^2 + dz^2) - B^2(t)d\psi^2, \tag{1}$$

where $B(t)$ and $R(t)$ are the scale factors.

The universe is assumed to be filled with distribution of matter represented by energy-momentum tensor of a perfect fluid

$$T_{\mu\nu} = (p+\rho)u_\mu u_\nu - p g_{\mu\nu}, \tag{2}$$

where ρ is the energy density of the cosmic matter and p is its pressure and u_μ is the five-velocity vector such that $u_\mu u^\mu = 1$.

The Einstein field equation with time-dependent cosmological and gravitational constants is given by

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G(t) T_{\mu\nu} + \Lambda(t) g_{\mu\nu}, \tag{3}$$

where $R_{\mu\nu}$ is the Ricci tensor, $G(t)$ and $\Lambda(t)$ being the variable gravitational and cosmological constants.

The divergence of (3), taking into account the Bianchi identity, gives

$$(8\pi G T_{\mu\nu} + \Lambda g_{\mu\nu})^{;\nu} = 0. \tag{4}$$

Equations (3) and (4) may be considered as the fundamental equations of gravity with G and Λ coupling parameters. Using comoving coordinates

$$u_\mu = (1, 0, 0, 0, 0), \tag{5}$$

in (2) and with line element (1), Einstein's field (3) by assuming $B(t) = R^n$ yields three independent equations

$$8\pi G(t)\rho = 3(n+1)\frac{\dot{R}^2}{R^2} - \Lambda(t), \tag{6}$$

$$8\pi G(t)p = -(n+2)\frac{\ddot{R}}{R} - (n^2+n+1)\frac{\dot{R}^2}{R^2} + \Lambda(t), \tag{7}$$

$$8\pi G(t)p = -3\left(\frac{\ddot{R}}{R} + \frac{\dot{R}^2}{R^2}\right) + \Lambda(t), \tag{8}$$

where dot denotes derivative with respective to t.

In uniform cosmology $G = G(t)$ and $\Lambda = \Lambda(t)$ so that the conservation (4) is given by

$$\dot{\rho} + (3+n)(\rho+p)H = -\left(\frac{\dot{G}}{G}\rho + \frac{\dot{\Lambda}}{8\pi G}\right). \tag{9}$$

The usual energy momentum conservation relation $T^{\mu\nu}_{;\nu} = 0$ leads to

$$\dot{\rho} + (3+n)(\rho+p)H = 0. \tag{10}$$

Therefore, (9) yields

$$\dot{\Lambda} = -8\pi \dot{G}\rho. \tag{11}$$

The field equations (6)–(8) can be written as

$$3(n+1)\ddot{R} = -8\pi G(t) R\left[(n+1)p + \rho - \frac{n\Lambda(t)}{8\pi G(t)}\right], \tag{12}$$

$$3(n+1)\dot{R}^2 = 8\pi G(t) R^2\left[\rho + \frac{\Lambda(t)}{8\pi G(t)}\right]. \tag{13}$$

Equations (12) and (13) can be rewritten in terms of the Hubble parameter $H=\dot{R}/R$ to give, respectively

$$\dot{H}+H^2=-\frac{8\pi}{3(n+1)}G(t)[(n+1)p+\rho]+\frac{n}{3(n+1)}\Lambda(t), \tag{14}$$

$$H^2=\frac{8\pi}{3(n+1)}G(t)\rho+\frac{1}{3(n+1)}\Lambda(t). \tag{15}$$

The system of equations (11), (14) and (15) may be solved by a physical assumption i.e. equation of state and from additional explicit assumption on H, $G(t)$ and $\Lambda(t)$ in terms of t or H which itself depends on cosmic time t. We take the equation of state

$$p=(\gamma-1)\rho, \tag{16}$$

where γ is an adiabatic parameter varying continuously with cosmological time so that in the course of its evolution the universe goes through a transition from an inflationary phase to radiation-dominated phase. Carvalho [30] assumed the functional form of γ depends on scale factor as

$$\gamma(R)=\frac{4}{3}\frac{A(\frac{R}{R_*})^2+(\frac{a}{2})(\frac{R}{R_*})^a}{A(\frac{R}{R_*})^2+(\frac{R}{R_*})^a}, \tag{17}$$

where A is a constant and a is free parameter related to the power of cosmic time and lies $0\le a<1$. Here R_* is certain reference value such that if $R\ll R_*$, inflationary phase of the evolution of the universe is obtained and for $R\gg R_*$, we have a radiation-dominated phase.

Using (16) into (14) we obtain

$$\dot{H}+H^2=-\frac{8\pi}{3(n+1)}G(t)[(n+1)\gamma-n]\rho+\frac{n}{3(n+1)}\Lambda(t). \tag{18}$$

Eliminating ρ between (11) and (18), we get

$$\dot{H}+H^2=\frac{1}{3(n+1)}[(n+1)\gamma-n]G(t)\frac{\dot{\Lambda}}{\dot{G}}+\frac{n}{3(n+1)}\Lambda(t). \tag{19}$$

The (19) we also rewrite as

$$HH'+\frac{H^2}{R}=\frac{1}{3(n+1)}[(n+1)\gamma-n]\frac{G(t)}{R}\frac{\Lambda'}{G'}+\frac{n}{3(n+1)}\frac{\Lambda(t)}{R}, \tag{20}$$

where a prime denotes differentiation with respect to R.

3 Solution of the Field Equations

Equation (20), involving H, G and Λ admits solutions for H and only if $\Lambda(t)$ and $G(t)$ are satisfied. According to LNH [23, 26], gravitational constant G varies linearly with Hubble parameter i.e. $G(t)$ decreases with age of the universe. The variation of $\Lambda(t)$ can resolve some of the standard model problems such as flatness, horizon, monopole etc. Like G, the constant $\Lambda(t)$ is a gravity coupling and both should therefore be treated on equal footing. A proper way in which G varies, is incorporated in Brans-Dicke theory [32]. We obtain the solution of (20) by taking certain assumptions on G and Λ.

3.1 Case (i)

We assume that

$$G(t) = \alpha H, \tag{21}$$

and

$$\Lambda(t) = \beta H^2, \tag{22}$$

where α and β are dimensionless positive constants.

Substituting the values of $G(t)$ and $\Lambda(t)$ from (21) and (22) into (20), we obtain

$$H^{'} + \left[1 + \frac{n\beta}{3(n+1)} - \frac{2\beta}{3}\gamma\right]\frac{H}{R} = 0. \tag{23}$$

Using the value of γ from (17) into (23) after integrating, we get

$$H = \frac{C}{R^{[1+\frac{n\beta}{3(n+1)}]}[A(\frac{R}{R_*})^2 + (\frac{R}{R_*})^a]^{-\frac{4\beta}{9}}}, \tag{24}$$

where C is the integration constant.

If $H = H_*$ for $R = R_*$, we have a relation between constant A and C, is given by

$$C = H_* R_*^{[1+\frac{n\beta}{3(n+1)}]}(1+A)^{-\frac{4\beta}{9}}. \tag{25}$$

Substitute this value in (24) and after integrating we get an expression for t in term of R is given by

$$H_* R_*^{[1+\frac{n\beta}{3(n+1)}]}(1+A)^{-\frac{4\beta}{9}} t = \int R^{\frac{n\beta}{3(n+1)}}\left[A\left(\frac{R}{R_*}\right)^2 + \left(\frac{R}{R_*}\right)^a\right]^{-\frac{4\beta}{9}} dR. \tag{26}$$

By defining $q = -(\frac{R\ddot{R}}{\dot{R}^2})$, it follows from (23) that during the course of evolution the deceleration parameter is given by

$$q = \frac{\beta}{3}\left(\frac{n}{(n+1)} - 2\gamma\right), \tag{27}$$

which is clearly depends upon R via γ. We solve (26) for two different early phases: *inflationary* and *radiation-dominated*.

3.1.1 Inflationary Phase

For inflationary phase ($R \ll R_*$), the second term on right hand side of integral in (26) dominates which gives the solution for scale factor R ($a \neq 0$) as,

$$R = R_*\left[\frac{H_*}{B_0(1+A)^{\frac{4\beta}{9}}} t\right]^{B_0}, \tag{28}$$

where

$$B_0 = \frac{1}{[1 - (\frac{4}{9}a - \frac{n}{3(n+1)})\beta]}.$$

Equation (28) shows that during inflation, the dimensions of the universe increase according to law

$$R \propto t^{B_0}, \tag{29}$$

which is the case of power-law of inflation. If $\beta = 0$ we see that the radius of the curvature increases linearly with age of the universe. Now we compute the Hubble parameter, gravitational constant, cosmological constant, and energy density to illustrate some observational predictions of the cosmological model of the universe.

From (28), we find the following solutions for physical parameters:

$$H = B_0 t^{-1}, \tag{30}$$

$$G = \alpha B_0 t^{-1}, \tag{31}$$

$$\Lambda = \beta {B_0}^2 t^{-2}, \tag{32}$$

and

$$\rho = \left(\frac{3(n+1) - \beta}{8\pi\alpha}\right) B_0 t^{-1}. \tag{33}$$

From the above solutions we observe that the energy density to be positive definite, we must have $0 < \beta < 3(n+1)$. The energy density tends to infinity as t tends to zero. The gravitational 'constant' G varies inversely as the age of universe, whereas cosmological 'constant' varies inversely as the square of the age of universe, which matches with their natural dimensions. Putting the limiting value $\gamma = \frac{2a}{3}$ in (27), the asymptotic value of deceleration parameter in the limit $\frac{R}{R_*} \ll 1$, is given by

$$q = \frac{1}{B_0} - 1. \tag{34}$$

3.1.2 Radiation-Dominated Phase

For radiation-dominated phase ($R \gg R_*$), the first term on right-hand side of the integral in (26) dominates which gives the solution for scale factor

$$R = R_* \left[\frac{1}{B_1}\left(\frac{A}{1+A}\right)^{\frac{4\beta}{9}} H_* t\right]^{B_1}, \tag{35}$$

where

$$B_1 = \frac{1}{\left[1 + \left(\frac{n}{3(n+1)} - \frac{8}{9}\right)\beta\right]}.$$

From (35) we observe that during radiation-dominated phase the dimension of the universe increase according to the law

$$R \propto t^{B_1}. \tag{36}$$

If $\beta = 0$, the radius of the universe increases linearly with cosmic time. From (35) we find the following solutions for Hubble parameter, gravitational constant, cosmological constant and energy density

$$H = B_1 t^{-1}, \tag{37}$$

$$G = \alpha\, B_1\, t^{-1}, \tag{38}$$

$$\Lambda = \beta\, B_1^2\, t^{-2}, \tag{39}$$

and

$$\rho = \frac{[3(n+1)-\beta]}{8\pi\alpha} B_1 t^{-1}. \tag{40}$$

respectively. For energy density to be positive, we must have $0 < \beta < 3(n+1)$. The energy density decreases with the cosmic time and tends to zero as t tends to infinity. The gravitational constant varies inversely with the cosmic time, whereas $\Lambda \propto t^{-2}$. Putting the limiting value $\gamma = \frac{4}{3}$ in (27) the asymptotic value of deceleration parameter in the limit $\frac{R}{R_*} > 1$, is given by

$$q = \frac{1}{B_1} - 1. \tag{41}$$

3.2 Case (ii)

We assume that

$$G = \frac{\alpha_1}{H}, \tag{42}$$

where α_1 is a positive constant.

Using (42) and (22) into (20), we obtain

$$H' + \left(1 + \frac{2\beta\gamma}{3} - \frac{n\beta}{n+1)}\right)\frac{H}{R} = 0, \tag{43}$$

which on integrating gives,

$$H = \frac{C_1}{R^{(1-\frac{n\beta}{n+1})}[A(\frac{R}{R_*})^2 + (\frac{R}{R_*})^a]^{\frac{4\beta}{9}}}, \tag{44}$$

where C_1 is the integration constant. An expression for t in terms of scale factor R is given by

$$H_* R_*^{(1-\frac{n\beta}{n+1})}(1+A)^{\frac{4\beta}{9}} t = \int R^{-\frac{n\beta}{n+1}} \left[A\left(\frac{R}{R_*}\right)^2 + \left(\frac{R}{R_*}\right)^a\right]^{\frac{4\beta}{9}} dR. \tag{45}$$

Using (45), we obtain the solutions for two different early phases of the universe: *inflationary and radiation-dominated.*

For **inflationary phase** we find the following solutions respectively,

$$R \propto t^{\frac{1}{\left(1+\frac{4\beta a}{9}-\frac{n\beta}{n+1}\right)}}, \tag{46}$$

$$H \propto t^{-1}, \tag{47}$$

$$G \propto t, \tag{48}$$

$$\Lambda \propto t^{-2}, \tag{49}$$

$$\rho \propto t^{-3}, \tag{50}$$

and for **radiation-dominated phase** we obtain the following solutions respectively,

$$R \propto t^{\frac{1}{\left(1+\frac{8\beta}{9}-\frac{n\beta}{n+1}\right)}} \tag{51}$$

$$H \propto t^{-1}, \tag{52}$$

$$G \propto t, \tag{53}$$

$$\Lambda \propto t^{-2}, \tag{54}$$

$$\rho \propto t^{-3}. \tag{55}$$

From the above solutions we observe that gravitational 'constant' increases with the cosmic time in both phases whereas cosmological 'constant' varies inversely as the square of the cosmic time. The energy density varies inversely as the cube of the cosmic time and hence tends to infinity as t tends to zero.

4 Conclusion

The existence of the extra dimension is a general feature in theories beyond the standard model in particle physics. It may manifest itself as source of energy in the ordinary three-space, such as "effective", dark energy or even "effective" dark matter. The geometrical structure and evolution pattern of extra dimension therefore may play an important role in cosmology. In this work, we have presented a solution of the Einstein field equations with variable Λ and G in the context of higher dimensional space time. We have consider Kaluza-Klein type cosmological model to study early evolution of the Universe as it goes from an inflationary phase to radiation dominated era. We have taken the "gamma-law" equation of state but with variable adiabatic parameter γ as the scale factor R and we have obtained cosmological solution, which may be important in describing the early universe. The behavior of the scale factor, cosmological constant, energy density and gravitational constant have been studied for two different phases: *inflationary and radiation*.

In case (i) it has been observed that the gravitational constant and energy density vary inversely with cosmic time whereas $\Lambda \propto t^{-2}$. We also observed that $\rho > 0$, for $\alpha \neq 0$ and β must be in the interval $0 < \beta < 3(n+1)$, which is also the condition for the expansion of the universe. In case (ii) we have obtained $G \propto t$ and $\Lambda \propto t^{-2}$, whereas energy density varies inversely as the cube of cosmic time for both *inflationary and radiation* dominated phases. The results show the possibility of increasing G has been observed by assuming $G \propto H^{-1}$. The expanding universe has singularity at $t = 0$. In both the cases the cosmological parameter retains the natural dimension with time i.e. $\Lambda \propto t^{-2}$. It is also compatible with Dirac's cosmology with varying cosmological constant (Lima and Carvalho [33]). In this way, unified description of early evolution of the universe is possible with variable gravitational and cosmological "constants" in the context of higher dimensional space time.

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